

THE IMPLICIT FUNCTION THEOREM FOR CONTINUOUS FUNCTIONS

CARLOS BIASI, CARLOS GUTIERREZ AND EDIVALDO L. DOS SANTOS

ABSTRACT. In the present paper we obtain a new homological version of the implicit function theorem and some versions of the Darboux theorem. Such results are proved for continuous maps on topological manifolds. As a consequence, some versions of these classic theorems are proved when we consider differentiable (not necessarily C^1) maps.

1. INTRODUCTION

This paper deals with a generalization of the classical Implicit Function Theorem. In this respect, C. Biasi and E. L. dos Santos proved a *homological version of the implicit function theorem* for continuous functions on general topological spaces which has interesting applications in the theory of topological groups. More specifically, Theorem 2.1 of [1] states that: “If X, Y, Z are Hausdorff spaces, with X locally connected, Y locally compact and $f : X \times Y \rightarrow Z$ such that $(f_{x_0})_* = (f|_{(x_0 \times Y)})_* : H_n(Y, Y - y_0) \rightarrow H_n(Z, Z - z_0)$ is a nontrivial homomorphism, for some natural $n > 0$, where $\{y_0\} = (f_{x_0})^{-1}(\{z_0\})$, then there exists an open neighborhood V of x_0 and a function $g : V \subset X \rightarrow Y$ satisfying the equation $f(x, g(x)) = z_0$, for each $x \in V$. Moreover, g is continuous at x_0 ”. This result establishes the existence of an implicit function g , however, the continuity of such function is guaranteed only at the point x_0 . In this paper, under little stronger assumptions, we can ensure the continuity of g at a neighborhood of x_0 . More precisely,

Theorem 4.1 *Let X be a locally pathwise connected Hausdorff space and let Y, Z be oriented connected topological manifolds of dimension n . Let $f : X \times Y \rightarrow Z$ be a continuous map such that, for all $x \in X$, the map $f_x : Y \rightarrow Z$ given by $f_x(y) = f(x, y)$ is open and discrete. Suppose that for some $(x_0, y_0) \in X \times Y$, $|\deg(f_{x_0}, y_0)| = 1$. Then there exists a neighborhood V of x_0 in X and a continuous function $g : V \rightarrow Y$ such that $f(x, g(x)) = f(x_0, y_0)$, for all $x \in V$.*

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As a consequence we obtain

Corollary 4.2 *Let X be a locally pathwise connected Hausdorff space. Let $U \subset \mathbb{R}^n$ be an open subset of \mathbb{R}^n and let $f : X \times U \rightarrow \mathbb{R}^n$ be a continuous map. Suppose that $f_x : U \rightarrow \mathbb{R}^n$ is a differentiable (not necessarily C^1) map without critical points, for each $x \in X$. Then there exist a neighborhood V of x_0 and a continuous function $g : V \rightarrow U$ such that $f(x, g(x)) = z_0$, for each $x \in V$.*

Corollary 4.3 *Let $I \times V \times W$ be an open neighborhood of (t_0, x_0, y_0) in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ and let $F : (I \times V \times W, (t_0, x_0, y_0)) \rightarrow (\mathbb{R}^m, 0)$ be a continuous map. Suppose that, for all $(t, x) \in I \times V$, the map $y \in W \rightarrow F(t, x, y)$ is differentiable (but not necessarily C^1) and without critical points. Then the differential equation*

$$F(t, x, x') = 0, \quad x(t_0) = x_0, \quad x'(t_0) = y_0$$

has a solution in some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.

We also prove the following versions of Darboux Theorem

Theorem 5.1 *Let M and N be oriented connected topological manifolds of dimension n and let $f : M \rightarrow N$ be a continuous map. Suppose that there exist x_0 and x_1 in M such that $\deg(f, x_0) < 0$ and $\deg(f, x_1) > 0$. Then there exists x_2 in M such that $\deg(f, x_2) = 0$*

Corollary 5.2 *Let M and N be oriented connected topological manifolds of dimension n and let $f : M \rightarrow N$ be a differentiable map. Suppose that there exist x_0 and x_1 in M such that $\det[f'(x_0)] < 0$ and $\det[f'(x_1)] > 0$. Then f has a critical point.*

A fundamental step to establish the versions of the Implicit Function Theorem and Darboux Theorem is Lemma 3.1 (Key Lemma).

2. PRELIMINARES

In this section we introduce some basic notions, notations and results that will be used throughout this paper. All considered singular homology groups will always have coefficients in \mathbb{Z} . By *dimension* we understand the usual topological dimension in the sense of [5].

In the following definitions, X and Y will be oriented connected topological manifolds of dimension $n \geq 1$ and $f : X \rightarrow Y$ will be a continuous map. The definition of orientation for topological manifolds can be found, for instance, in [4].

Definition 2.1. A map $f : X \rightarrow Y$ is said to be *discrete* at a point x_0 , if there exists a neighborhood V of x_0 such that $f(x) \neq f(x_0)$, for any $x \in V - x_0$, that is, $f^{-1}(f(x_0)) \cap (V - x_0) = \emptyset$.

Definition 2.2. Let $y \in Y$ such that $L_y = f^{-1}(y)$ is a compact subset of X . Let $\alpha_{L_y} \in H_n(X, X - L_y)$ and $\beta_y \in H_n(Y, Y - y)$ be the orientation classes along L_y at y ; respectively. There exists an integer number $\deg(f, y)$ such that $f_*(\alpha_{L_y}) = \deg(f, y) \cdot \beta_y$, where $f_* : H_n(X, X - L_y) \rightarrow H_n(Y, Y - \{y\})$ is the homomorphism induced by f . The number $\deg(f, y)$ is called *degree of f at y* .

Definition 2.3. Let $f : X \rightarrow Y$ be a *discrete map* at a point x_0 and let us denote by $y_0 = f(x_0)$. Consider $\alpha_{x_0} \in H_n(V, V - x_0)$ and $\beta_{y_0} \in H_n(Y, Y - y_0)$ the orientation classes at x_0 and y_0 , respectively. There exists an integer number $\deg(f, x_0)$ such that $f_*(\alpha_{x_0}) = \deg(f, x_0) \cdot \beta_{y_0}$, where the homomorphism $f_* : H_n(V, V - x_0) \rightarrow H_n(Y, Y - y_0)$ is induced by f . The number $\deg(f, x_0)$ is called *local degree of f at x_0* .

Definition 2.4. Suppose that $f : X \rightarrow Y$ is not necessarily a discrete map. Define

$$\deg(f, x) = \begin{cases} 0, & \text{if } f \text{ is not discrete at } x. \\ \deg(f, x), & \text{if } f \text{ is discrete at } x, \text{ in the sense of Definition 2.3.} \end{cases}$$

The proof of the following two propositions can be found in [4].

Proposition 2.5. Let X and Y be oriented connected topological manifolds of dimension $n \geq 1$ and let $f : X \rightarrow Y$ be a continuous map. Consider $y \in Y$ such that $f^{-1}(y_0)$ is finite. Then,

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \deg(f, x).$$

Proposition 2.6. Let X and Y be oriented connected topological manifolds of dimension $n \geq 1$ and $f : X \rightarrow Y$ be a continuous map. Let us consider a compact connected subset K of Y such that $L_K = f^{-1}(K)$ is compact. Then, $\deg(f, y)$ does not depend of $y \in K$.

As an immediate consequence of Proposition 2.6 we have the following:

Corollary 2.7. Let X and Y be oriented connected topological manifolds of dimension $n \geq 1$ and $f : X \rightarrow Y$ be a proper continuous map. Then, for all $y \in Y$, $\deg(f, y) = \deg(f)$.

In [7], Väisälä proved the following version of the Černavskii's theorem (see [2, 3]):

Theorem 2.8. Let X and Y be topological manifold of dimension n . Suppose that $f : X \rightarrow Y$ is an open and discrete map. Then $\dim(f(B_f)) \leq n - 2$ and $\dim(B_f) \leq n - 2$, where B_f denotes the set of points x of X such that f is not a local homeomorphism at x .

3. PERSISTENCE OF THE SIGN OF $\deg(f, x)$

The following lemma will be fundamental in the proof of the versions of the Implicit Function Theorem and Darboux Theorem.

Lemma 3.1 (Key Lemma). *Let X and Y be oriented connected topological manifolds of dimension n . Suppose that $f : X \rightarrow Y$ is an open and discrete map. Then, for any $x \in X$, one has that $\deg(f, x) \neq 0$; moreover, $\deg(f, x)$ has always the same sign.*

Proof. It follows from Theorem 2.8 that $\dim B_f \leq n - 2$, then $X - B_f$ is connected. Since f is a local homeomorphism on $X - B_f$, one has that

$$(3.1) \quad \deg(f, x) = c, \forall x \in X - B_f, \text{ where either } c = 1 \text{ or } c = -1.$$

Let $x_0 \in B_f$ and denote by $y_0 = f(x_0)$. Since f is a discrete map and X is locally compact, we can choose an open neighborhood V of x_0 such that \overline{V} is compact and $f(x) \neq f(x_0)$ for all $x \in \overline{V} - x_0$. Then, there exists an open neighborhood W of $y_0 = f(x_0)$ such that \overline{W} is a compact and connected subset of Y and $(f|_{\overline{V}})^{-1}(\overline{W}) = (f|_V)^{-1}(\overline{W}) \subset V$. Therefore, $(f|_V)^{-1}(\overline{W})$ is compact and it follows from Proposition 2.6 that

$$(3.2) \quad \deg(f|_V, y) = \deg(f|_V, y_0), \forall y \in \overline{W}$$

On the other hand, since $U = (f|_V)^{-1}(W)$ is an open set in X and f is a open map, we have that $f|_V(U) \subset \overline{W}$ is an open set in Y and since $Y - f(B_f)$ is dense in Y (recall that $\dim(f(B_f)) \leq n - 2$ by Theorem 2.8), there exists $y_1 \in f|_V(U) - f(B_f)$. Therefore, it follows from (3.2) that $\deg(f|_V, y_1) = \deg(f|_V, y_0)$. Let $f^{-1}(y_1) = \{x_1, \dots, x_k\}$ in $X - B_f$. Then, by Proposition 2.5 we have that

$$(3.3) \quad \deg(f, x_0) = \deg(f|_V, y_0) = \deg(f|_V, y_1) = \sum_{x_i \in f^{-1}(y_1)} \deg(f|_V, x_i).$$

On the other hand, since $x_i \in X - B_f$ for $i = 1, \dots, k$, by formula (3.1), $\deg(f, x_i) = c$ where either $c = 1$ or $c = -1$. Then, it follows from (3.3) that $\deg(f, x_0) = kc$ and therefore $\deg(f, x)$ has always the same sign, for any $x \in X$. \square

4. THE IMPLICIT FUNCTION THEOREM

In this section we shall show a new homological version of the Implicit Function Theorem.

Theorem 4.1 (Implicit Function Theorem). *Let X be a locally pathwise connected Hausdorff space and let Y, Z be oriented connected topological manifolds of dimension n . Let $f : X \times Y \rightarrow Z$ be a continuous map such that, for all $x \in X$, the map*

$f_x : Y \rightarrow Z$ given by $f_x(y) = f(x, y)$ is open and discrete. Suppose that for some $(x_0, y_0) \in X \times Y$, $|\deg(f_{x_0}, y_0)| = 1$. Then there exists a neighborhood V of x_0 in X and a continuous function $g : V \rightarrow Y$ such that $f(x, g(x)) = f(x_0, y_0)$, for all $x \in V$.

Proof. Let $z_0 = f(x_0, y_0)$. Since Y is locally compact and f_{x_0} is a discrete map, we can choose a compact neighborhood $W \subset Y$ of $y_0 \in Y$ satisfying

$$(4.1) \quad (f_{x_0})^{-1}(z_0) \cap W = y_0.$$

We will first show that for any compact neighborhood $K \subset \text{Int}(W)$ containing y_0 , there exists a neighborhood V of x_0 such that

$$(4.2) \quad (f_{x|_W})^{-1}(\{z_0\}) \subseteq K, \quad \forall x \in V.$$

In fact, suppose that for each neighborhood V of x_0 there exists (x_V, y_V) in $V \times (W - K)$ such that $f(x_V, y_V) = z_0$. Let us consider a generalized sequence, called a net, $((x_V, y_V))_{V \in \mathcal{V}}$, where \mathcal{V} is the collection of all the neighborhoods of x_0 , partially ordered by reverse inclusion (for details see [6, pg.187 and 188]). Therefore, $\lim x_V = x_0$ and since (y_V) is a net contained in the compact subset $W - \text{Int}(K)$, there exists $y_1 \in W - \text{Int}(K)$ which is a limit point of some convergent subnet of (y_V) . Hence, (x_0, y_1) is a limit point of some subnet of (x_V, y_V) . Since f is a continuous map, one has $f(x_0, y_1) = z_0$ which implies that $y_1 = y_0$, contradicting the fact that $y_1 \in W - \text{Int}(K)$. Therefore, there exists a neighborhood V of x_0 satisfying (4.2).

Choose a compact neighborhood $K \subset \text{Int}(W)$ of y_0 . It follows from (4.2) that for each $x \in V$, the map of pairs $f_x : (W, W - K) \rightarrow (Z, Z - z_0)$ is well defined. Since X is locally pathwise connected, we can assume that V is a pathwise connected neighborhood of x_0 and therefore, for each x in V , there exists a path α in V joining x_0 to x . We define the homotopy of pairs $H : (I \times W, I \times (W - K)) \rightarrow (Z, Z - z_0)$ given by

$$(4.3) \quad H(t, y) = f(\alpha(t), y) = f_{\alpha(t)}(y), \quad \forall (t, y) \in I \times W.$$

Since W is compact and H is a proper homotopy between $f_{x_0}|_W$ and $f_x|_W$, we obtain $\deg(f_x|_W, z_0) = \deg(f_{x_0}|_W, z_0)$. Now, as $\deg(f_{x_0}|_W, z_0) = \deg(f_{x_0}, y_0)$ and as, by hypotheses, $|\deg(f_{x_0}, y_0)| = 1$ we obtain

$$(4.4) \quad |\deg(f_x|_W, z_0)| = |\deg(f_{x_0}|_W, z_0)| = |\deg(f_{x_0}, y_0)| = 1.$$

Since f_x is open and discrete, it follows from Lemma 3.1 that $\deg(f_x, y) \neq 0$, for any $y \in Y$, and $\deg(f_x, y)$ has always the same sign. Thus, if $(f_x|_W)^{-1}(z_0) =$

$\{y_1, \dots, y_k\} \subset W$ with $k \geq 2$, we have that

$$(4.5) \quad |\deg(f_x|_W, z_0)| = \left| \sum_{i=1}^k \deg(f_x, y_i) \right| = \sum_{i=1}^k |\deg(f_x, y_i)| > 1,$$

which contradicts (4.4). Therefore, for each $x \in V$ there exists a unique $y \in K \subset W$ such that $(f_x|_W)^{-1}(z_0) = y$; in other words, for each $x \in V$ there exists a unique $y = g(x) \in K$ such that $f_x(g(x)) = f(x, g(x)) = z_0$ for each $x \in V$.

We will show that the map $g : V \rightarrow K \subset Y$ is continuous. Let A be a neighborhood of $y = g(x)$ such that $A \subset K$. Let us assume that for any neighborhood U of x , there exists x_U in U such that $g(x_U)$ belongs to the compact subset $K - A$. Let \bar{y} in $K - A$ be the limit point of the some convergent subnet of $(g(x_U))$. Thus, (x, \bar{y}) is the limit point of the some convergent subnet of $(x_U, g(x_U))$. Since f is continuous and g is given implicitly by the equation $f(x_U, g(x_U)) = z_0$ we have that $f(x, \bar{y}) = z_0$, which implies that $y = \bar{y}$, contradicting the fact that $\bar{y} \in K - A$. \square

Corollary 4.2. *Let X be a locally pathwise connected Hausdorff space. Let $U \subset \mathbb{R}^n$ be an open subset of \mathbb{R}^n and let $f : X \times U \rightarrow \mathbb{R}^n$ be a continuous map. Suppose that $f_x : U \rightarrow \mathbb{R}^n$ is a differentiable (not necessarily C^1) map without critical points, for each $x \in X$. Then there exist a neighborhood V of x_0 and a continuous function $g : V \rightarrow U$ such that $f(x, g(x)) = z_0$, for each $x \in V$.*

Proof. Since $f_x : U \rightarrow \mathbb{R}^n$ is a differentiable (not necessarily C^1) map without critical points, we have that f_x is an open and discrete map, for each x in X . Therefore, the result follows from Theorem 4.1 \square

Corollary 4.3. *Let $I \times V \times W$ be an open neighborhood of (t_0, x_0, y_0) in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ and let $F : (I \times V \times W, (t_0, x_0, y_0)) \rightarrow (\mathbb{R}^m, 0)$ be a continuous map. Suppose that, for all $(t, x) \in I \times V$, the map $y \in W \rightarrow F(t, x, y)$ is differentiable (but not necessarily C^1) and without critical points. Then the differential equation*

$$F(t, x, x') = 0, \quad x(t_0) = x_0, \quad x'(t_0) = y_0$$

has a solution in some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.

Proof. By Corollary 4.2, we have that there exists a neighborhood $(t_0 - \varepsilon_1, t_0 + \varepsilon_1) \times V_1$ of (t_0, x_0) and a function $g : (t_0 - \varepsilon_1, t_0 + \varepsilon_1) \times V_1 \rightarrow W$ such that

$$F(t, x, g(t, x)) = 0$$

By Peano Theorem, there exists a solution $\varphi : (t_0 - \varepsilon, t_0 + \varepsilon)$ of the differential equation

$$x' = g(t, x), \quad x(t_0) = x_0, \quad x'(t_0) = y_0.$$

This implies the Corollary. \square

5. GENERALIZATIONS OF THE DARBOUX THEOREM

The classical Darboux theorem states that if $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable map which has the property that $f'(a) < 0$ and $f'(b) > 0$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

In order to obtain some versions of the Darboux theorem we apply the results previously obtained.

Theorem 5.1 (A homological version of the Darboux theorem). *Let M and N be oriented connected topological manifolds of dimension n and let $f : M \rightarrow N$ be a continuous map. Suppose that there exist x_0 and x_1 in M such that $\deg(f, x_0) < 0$ and $\deg(f, x_1) > 0$, then there exists x_2 in M such that $\deg(f, x_2) = 0$*

Proof. If f is not a *discrete map* at some $x \in M$, it follows from Definition 2.4 that $\deg(f, x) = 0$. In this way, suppose that f is a *discrete map* at x , for every $x \in M$. Therefore, if $\deg(f, x) \neq 0$ for any $x \in M$, then f is also an open map and from Lemma 3.1 we conclude that $\deg(f, x)$ has always the same sign, for any $x \in M$, which is a contradiction. \square

The following theorems are differentiable versions of the Darboux theorem. Let us observe that when f is of class C^1 , these results are trivial.

Corollary 5.2. *Let M and N be oriented connected topological manifolds of dimension n and let $f : M \rightarrow N$ be a differentiable map. Suppose that there exist x_0 and x_1 in M such that $\det[f'(x_0)] < 0$ and $\det[f'(x_1)] > 0$. Then f has a critical point.*

Proof. Suppose that f does not have critical points. In this case, one has that for any $x \in M$, $|\deg(f, x)| = 1 \neq 0$. Thus, f is an open and *discrete map* and it follows from Lemma 3.1 that $\deg(f, x)$ has always the same sign, which is a contradiction. \square

Corollary 5.3. *Consider $f, g : M \rightarrow \mathbb{R}^n$ differentiable maps, where M is an oriented connected topological manifold of dimension n . Suppose that there exist $\alpha \in \mathbb{R}$ and $x_0, x_1 \in M^n$ such that*

$$\det[f'(x_0) - \alpha g'(x_0)] < 0 \quad \text{and} \quad \det[f'(x_1) - \alpha g'(x_1)] > 0.$$

Then, there exists $x_2 \in M$ such that $\det[f'(x_2) - \alpha g'(x_2)]$ is equal to zero.

Proof. It suffices to apply Theorem 5.2 for the map $h = f - \alpha g$. \square

As a direct consequence of Theorem 5.3 one has the following version of the classical Darboux theorem for differentiable maps from \mathbb{R}^n into \mathbb{R}^n .

Corollary 5.4. (Differentiable Darboux theorem) *Let $f : U \rightarrow \mathbb{R}^n$ be a differentiable map, where U is an open connected subset of \mathbb{R}^n . Suppose that there exist $\alpha \in \mathbb{R}$ and $x_0, x_1 \in U$ such that $\det[f'(x_0) - \alpha I] < 0$ and $\det[f'(x_1) - \alpha I] > 0$. Then, there exists $x_2 \in U$ such that $\det[f'(x_2) - \alpha I]$ is equal to zero (i.e. α is an eigenvalue of $f'(x_2)$).* \square

Now consider f and U under the same assumptions of Corollary 5.4 and let us denote by $p_0(\lambda) = \det[f'(x_0) - \lambda I]$ and by $p_1(\lambda) = \det[f'(x_1) - \lambda I]$. Let n_0 and n_1 natural numbers. In these conditions, we prove the following

Corollary 5.5. *Let $x_0, x_1 \in U$ and $\alpha \in \mathbb{R}$ such that $p_0(\lambda) = q_0(\lambda) \cdot (\alpha - \lambda)^{n_0}$ and $p_1(\lambda) = q_1(\lambda)(\alpha - \lambda)^{n_1}$, where $q_0(\lambda)$ and $q_1(\lambda)$ are not null polynomials. Suppose that n_0 is odd and n_1 is even. If $q_0(\alpha)q_1(\alpha) > 0$ then there exists $\delta > 0$ satisfying the following condition: for each $\lambda \in (\alpha, \alpha + \delta)$ there exists $x_2 = x_2(\lambda)$ such that $\det[f'(x_2) - \lambda I]$ is equal to zero.*

Proof. Since n_0 is odd and n_1 is even, one has that, for $\lambda > \alpha$ close to α , the polynomials $p_0(\lambda)$ and $p_1(\lambda)$ have different signs. It follows from Corollary 5.4 that if $\delta > 0$ is small enough and $\lambda \in (\alpha, \alpha + \delta)$, there exists $x_2 = x_2(\lambda)$ such that $p_2(\lambda) = \det[f'(x_2) - \lambda I]$ is equal to zero. \square

Remark 5.6. Theorem 5.5 remains the same in the case that $q_0(\alpha)q_1(\alpha) < 0$ and $\lambda \in (\alpha - \delta, \alpha)$.

REFERENCES

- [1] Biasi, C., dos Santos, E.L., *A homological version of the implicit function theorem*, Semigroup Forum, (**72**) n.3 , 353-361, 2006.
- [2] Černavskii, A. V., *Finite-to-one open mappings of manifolds*, Amer. Math. Soc. Translations, Series (2) 100, 253-267, translation of Math. Sbornik, (**65**) (107), n. 3, 357-369, 1964.
- [3] Černavskii, A. V., *Addendum to the paper "Finite-to-one open mappings of manifolds"*, Amer. Math. Soc. Translations, Series (2) 100, 269-270, translation of Math. Sbornik, (**66**) (108), n. 3, 471-472, 1965.
- [4] A. DOLD, *Lectures on Algebraic Topology* , Die Grundlehren der mathematischen Wissenschaften, Band 200, Springer - Verlag, 1972.
- [5] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Math. Series, Vol. 4, Princeton Univ. Press, Princeton, 1948.
- [6] J. R. Munkres, *Topology, a first course*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1975.
- [7] Väisälä, J., *Discrete open mappings on manifolds*, Annales Acad. Sci. Fenn. Ser. AI , n. **392**, 1966.

UNIVERSIDADE DE SÃO PAULO, DEPARTAMENTO DE MATEMÁTICA-ICMC, CAIXA POSTAL 668,
13560-970, SÃO CARLOS SP, BRAZIL

E-mail address: `biasi@icmc.usp.br`

E-mail address: `gutp@icmc.usp.br`

EDIVALDO L. DOS SANTOS

UNIVERSIDADE FEDERAL DE SÃO CARLOS, DEPARTAMENTO DE MATEMÁTICA CAIXA POSTAL
676, 13565-905, SÃO CARLOS SP, BRAZIL

E-mail address: `edivaldo@dm.ufscar.br`